1 Coverings

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1.1 Definitions and Examples

1.1 Definition Let $v \geq k \geq t$. A t-(v, k, λ) *covering* is a pair $(X, \mathcal{B})$, where X is a v-set of elements (*points*) and $\mathcal{B}$ is a collection of k-subsets (*blocks*) of X, such that every t-subset of points occurs in at least λ blocks in $\mathcal{B}$. Repeated blocks in $\mathcal{B}$ are permitted.

1.2 Definition The *covering number* $C_\lambda(v, k, t)$ is the minimum number of blocks in any t-(v, k, λ) covering. A t-(v, k, λ) covering $(X, \mathcal{B})$ is *optimal* if $|\mathcal{B}| = C_\lambda(v, k, t)$. If $\lambda = 1$, then write $C(v, k, t)$ for $C_1(v, k, t)$.

1.3 Examples Optimal coverings for certain parameter sets t-(v, k, λ).

t-(v, k, λ)	Covering
2-$(5, 3, 1)$	{1,2,3}, {1,4,5}, {2,3,4}, {2,3,5}
2-$(6, 3, 1)$	{0+i,1+i,3+i} modulo 6
2-$(8, 3, 1)$	{0+i,1+i,3+i} modulo 7, {0,1,∞}, {2,3,∞}, {4,5,∞}, {5,6,∞}
2-$(6, 4, 1)$	{1,2,3,4}, {1,2,5,6}, {3,4,5,6}
2-$(9, 4, 1)$	{1,2,3,4}, {1,2,5,6}, {1,7,8,9}, {2,4,6,8}, {2,7,8,9}, {3,5,8,9}, {3,6,7,9}, {4,5,7,9}

1.4 Remark The survey paper by Mills and Mullin [7] covers much of the material in this section, and gives extensive references. The web site [4] contains current bounds, and gives references to some of the more recent results.

1.2 Equivalent Combinatorial Objects

1.5 Theorem A t-(v, k, λ) covering with $\lambda\binom{v}{t}/\binom{k}{t}$ blocks is equivalent to a t-(v, k, λ) design or a Steiner system $S_\lambda(t, k, v)$ (possibly containing repeated blocks).

1.6 Definition Let $v \geq m \geq k$. A (v, m, k) *Turán design* is a pair $(X, \mathcal{B})$, where X is a v-set of elements (*points*) and $\mathcal{B}$ is a collection of k-subsets (*blocks*) of X, such that every m-subset of points is a superset of at least one block $B \in \mathcal{B}$.

1.7 Definition The *Turán number* $T(v, m, k)$ is the minimum number of blocks in any (v, m, k) Turán design.

1.8 Theorem $(X, \mathcal{B})$ is a (v, m, k) *Turán design* if and only if $(X, \{X \backslash B : B \in \mathcal{B}\})$ is a $(v - m)$-$(v, v - k, 1)$ covering.

1.9 Corollary $T(v, m, k) = C(v, v - k, v - m)$.

1.10 Definition An *(n,u,v,d) constant-weight covering code* is a code of length n, constant weight u, such that every word with weight v is within Hamming distance d of at least one codeword. $K(n, u, v, d)$ is the minimum size of such a code.

1.11 Theorem For $u - v \geq 0$, a $(n, u, v, u - v)$ constant-weight covering code is a (n, u, v) covering design.

1.12 Corollary For $u - v \geq 0$,

$$K(n, u, v, u - v) = C(n, u, v).$$

1.13 Definition An (n, k, p, t)*-lottery scheme* is a set of k-element subsets (*blocks*) of an n-set such that each p-subset intersects some block in at least t elements.

1.14 Theorem A (v, k, t, t)-lottery scheme is a t-$(v, k, 1)$ covering design.

1.15 Definition A *quorum system* is a pair $(X, \mathcal{A})$, where X is a v-set of elements, and $\mathcal{A}$ is a collection of subsets (*quorums*) of X such that any two quorums in $\mathcal{A}$ have a nonempty intersection.

1.16 Remark Quorum systems are used to maintain consistency in distributed systems. Connections between quorum systems and coverings are given in [3].

1.17 Definition A *directed* $t - (v, k, \lambda)$ *covering* is a pair $(X, \mathcal{B})$, where X is a v-set of elements, and $\mathcal{B}$ is a collection of *ordered* subsets of X such that every ordered t-subset of X occurs, in the same order, at least λ times.

1.18 Remark A directed $t-(v, k, \lambda)$ covering is a standard $t-(v, k, t!\,\lambda)$ covering. The size of a directed $t - (v, k, \lambda)$ covering is denoted $DC_\lambda(v, k, t)$. See [1] for recent results on these numbers.

1.3 Lower Bounds

1.19 Theorem (*Schönheim bound*) $C_\lambda(v, k, t) \geq \lceil v\, C_\lambda(v-1, k-1, t-1)/k \rceil$. Iterating this bound yields $C_\lambda(v, k, t) \geq L_\lambda(v, k, t)$, where

$$L_\lambda(v, k, t) = \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \cdots \left\lceil \frac{\lambda(v-t+1)}{k-t+1} \right\rceil \right\rceil \right\rceil .$$

Write $L(v, k, t)$ for $L_1(v, k, t)$.

1.20 Theorem (Hanani) If $\lambda(v-1) \equiv 0 \pmod{k-1}$ and $\lambda v(v-1) \equiv 1 \pmod{k}$, then

$$C_\lambda(v, k, 2) \geq L_\lambda(v, k, 2) + 1.$$

1.21 Remark Let $B_\lambda(v, k, t)$ be the lower bound implied by Theorems 1.19 and 1.20, which is either $L_\lambda(v, k, t)$ or $L_\lambda(v, k, t) + 1$. Write $B(v, k, t)$ for $B_1(v, k, t)$.

1.22 Theorem (Caro and Yuster [2]) For any k there is a $v_0 = v_0(k)$ such that $C(v, k, 2) = B(v, k, 2)$ for all $v > v_0$.

1.23 Table Aside from the Schönheim bound, most lower bound results in the literature are for individual covering numbers, and typically require analysis of many cases or extensive computer searches. This table gives some recent results, all for $\lambda = 1$. References are given in [4]. Values known to be exact are in **bold**.

v	k	t	lower bound	v	k	t	lower bound	v	k	t	lower bound
19	6	2	**15**	19	13	4	**11**	15	11	6	**21**
28	9	2	**14**	11	6	5	96	16	12	6	**19**
41	13	2	**14**	11	7	5	33	17	13	6	**17**
14	7	3	**15**	13	9	5	**19**	21	16	6	**17**
13	8	3	**10**	16	12	5	**12**	12	8	7	**126**
15	9	3	**10**	18	13	5	**15**	13	10	7	**30**
17	10	3	**11**	19	14	5	**14**	18	14	7	**24**
10	5	4	**51**	21	16	5	**12**	13	9	8	**185**
16	11	4	**12**	11	7	6	**84**	14	11	8	**40**
18	12	4	**12**	12	7	6	165	20	16	8	**26**

1.4 Determination of Covering Numbers

1.24 Theorem $C_\lambda(v,3,2) = B_\lambda(v,3,2)$.

1.25 Theorem $C(v,4,2) = L(v,4,2)+\epsilon$, where

$$\epsilon = \begin{cases} 1 & \text{if } v = 7, 9 \text{ or } 10 \\ 2 & \text{if } v = 19 \\ 0 & \text{otherwise.} \end{cases}$$

1.26 Theorem If $\lambda > 1$, then $C_\lambda(v,4,2) = L_\lambda(v,4,2)$.

1.27 Theorem $C(v,4,3) = L(v,4,3)$ except for $v = 7$ and possible exceptions of $v = 12k+7$ with $k \in \{1,2,3,4,5,7,8,9,10,11,12,16,21,23,25,29\}$.

1.28 Theorem $C(v,5,2) = B(v,5,2)$ except possibly when

1. $v = 15$,
2. $v \equiv 0 \pmod 4$, $v \le 280$
3. $v \equiv 9 \pmod{20}$, $v \le 429$,
4. $v \equiv 17 \pmod{20}$, $v \le 377$,
5. $v \equiv 13 \pmod{20}$, $v \in \{13,53,73\}$.

1.29 Theorem For $\lambda > 1$, $C_\lambda(v,5,2) = B_\lambda(v,5,2)$, except possibly when

1. $\lambda = 2$ and $v \in \{9,13,15,53,63,73,83\}$,
2. $\lambda \equiv 13 \pmod{20}$ and $v = 44$,
3. $\lambda = 17$ and $v = 44$.

1.30 Remark Theorem 1.29 is a very recent result of Bluskov and Greig. The only cases with $\lambda > 1$ where $C_\lambda(v,5,2)$ is known to be greater than $B_\lambda(v,5,2)$ is when $\lambda = 2$ and $v \in \{9,13,15\}$.

1.31 Theorem The values $C(v,k,2)$ are known in the following cases:

1. $C(v,k,2) = 3$ for $1 < v/k \le 3/2$;
2. $C(v,k,2) = 4$ for $3/2 < v/k \le 5/3$;
3. $C(v,k,2) = 5$ for $5/3 < v/k \le 9/5$;
4. $C(v,k,2) = 6$ for $9/5 < v/k \le 2$;
5. $C(v,k,2) = 7$ for $2 < v/k \le 7/3$, except when $3v = 7k-1$;
6. $C(v,k,2) = 8$ for $7/3 < v/k \le 12/5$, except when $12k-5v = 0,1$ and $v-k$ is odd;
7. $C(v,k,2) = 9$ for $12/5 < v/k \le 5/2$, except when $2v = 5k$ and $v-k$ is odd;

8. $C(v,k,2) = 10$ for $5/2 < v/k \le 8/3$, except when $8k - 3v \in \{0,1\}$, $v - k$ is odd, and $k > 2$;
9. $C(v,k,2) = 11$ for $8/3 < v/k \le 14/5$, except when $14k - 5v \in \{0,1\}$, $v - k$ is odd, and $k > 4$;
10. $C(v,k,2) = 12$ for $14/5 < v/k \le 3$, except when $v = 3k$, $k \not\equiv 0 \pmod 3$, and $k \not\equiv 0 \pmod 4$.
11. $C(v,k,2) = 13$ for $3 < v/k \le 13/4$, except for
    (a) $C(13r+2, 4r+1, 2) = 14, r \ge 2$,
    (b) $C(13r+3, 4r+1, 2) = 14, r \ge 2$,
    (c) $C(13r+6, 4r+2, 2) = 14, r \ge 2$,
    (d) $C(19,6,2) = 15$,
    (e) $C(16,5,2) = 15$.

1.32 Remark The exceptional cases are all known, and one block larger. The result on 13 blocks is recent, and due to Greig, Li, and van Rees.

1.33 Table Upper bounds on $C(v,k,2)$ for $v \le 32$ and $k \le 16$. Values known to be exact are in **bold**. All other values are one more than the lower bound.

$t = 2$

$v\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	**1**													
4	**3**	**1**												
5	**4**	**3**	**1**											
6	**6**	**3**	**3**	**1**										
7	**7**	**5**	**3**	**3**	**1**									
8	**11**	**6**	**4**	**3**	**3**	**1**								
9	**12**	**8**	**5**	**3**	**3**	**3**	**1**							
10	**17**	**9**	**6**	**4**	**3**	**3**	**3**	**1**						
11	**19**	**11**	**7**	**6**	**4**	**3**	**3**	**3**	**1**					
12	**24**	**12**	**9**	**6**	**5**	**3**	**3**	**3**	**3**	**1**				
13	**26**	**13**	**10**	**7**	**6**	**4**	**3**	**3**	**3**	**3**	**1**			
14	**33**	**18**	**12**	**7**	**6**	**5**	**4**	**3**	**3**	**3**	**3**	**1**		
15	**35**	**19**	**13**	**10**	**7**	**6**	**4**	**3**	**3**	**3**	**3**	**3**	**1**	
16	**43**	**20**	**15**	**10**	**8**	**6**	**5**	**4**	**3**	**3**	**3**	**3**	**3**	**1**
17	**46**	**26**	**16**	**12**	**9**	**7**	**6**	**5**	**4**	**3**	**3**	**3**	**3**	**3**
18	**54**	**27**	**18**	**12**	**10**	**7**	**6**	**5**	**4**	**3**	**3**	**3**	**3**	**3**
19	**57**	**31**	**19**	**15**	**11**	**9**	**7**	**6**	**5**	**4**	**3**	**3**	**3**	**3**
20	**67**	**35**	**21**	**16**	**12**	**9**	**7**	**6**	**6**	**4**	**4**	**3**	**3**	**3**
21	**70**	**37**	**21**	**17**	**13**	**11**	**7**	**7**	**6**	**5**	**4**	**3**	**3**	**3**
22	**81**	**39**	**27**	**19**	**13**	**11**	**9**	**7**	**6**	**6**	**5**	**4**	**3**	**3**
23	**85**	**46**	**28**	21	16	**12**	**10**	**8**	**7**	**6**	**5**	**4**	**4**	**3**
24	**96**	**48**	**30**	22	**17**	**12**	**11**	**8**	**7**	**6**	**6**	**5**	**4**	**3**
25	**100**	**50**	**30**	**23**	**18**	**13**	**11**	**10**	**7**	**7**	**6**	**5**	**4**	**4**
26	**113**	**59**	**37**	**24**	**19**	**13**	**12**	**10**	**8**	**7**	**6**	**6**	**5**	**4**
27	**117**	**61**	**38**	**27**	**20**	17	**12**	**11**	**9**	**7**	**7**	**6**	**5**	**5**
28	**131**	**63**	**40**	**28**	22	**17**	**14**	**11**	**10**	**7**	**7**	**6**	**6**	**5**
29	**136**	**73**	43	30	23	**18**	**14**	**12**	**10**	**9**	**7**	**7**	**6**	**6**
30	**150**	**75**	**48**	**31**	25	**19**	**15**	**13**	**11**	**9**	**8**	**7**	**6**	**6**
31	**155**	**78**	**50**	**31**	**26**	**20**	17	**13**	**12**	**10**	**8**	**7**	**7**	**6**
32	**171**	**88**	**52**	**38**	**28**	**20**	18	**14**	**12**	**10**	**9**	**7**	**7**	**6**

1.34 Theorem (Mills) The values $C(v,k,3)$ are known in the following cases:

1. $C(v,k,3) = 4$ for $1 < v/k \le 4/3$;
2. $C(v,k,3) = 5$ for $4/3 < v/k \le 7/5$;

3. $C(v,k,3) = 6$ for $7/5 < v/k \le 3/2$, except when $2v = 3k$ and v is odd;
4. $C(v,k,3) = 7$ for $3/2 < v/k \le 17/11$, except when $11v = 17k - 1$;
5. $C(v,k,3) = 8$ for $17/11 < v/k \le 8/5$, except when $5v = 8k - 1$ and $k > 7$.

1.35 Table Upper bounds on $C(v,k,3)$ for $v \le 32$ and $k \le 16$. Values known to be exact are in **bold**.

$t = 3$

$v \backslash k$	4	5	6	7	8	9	10	11	12	13	14	15	16
4	**1**												
5	**4**	**1**											
6	**6**	**4**	**1**										
7	**12**	**5**	**4**	**1**									
8	**14**	**8**	**4**	**4**	**1**								
9	**25**	**12**	**7**	**4**	**4**	**1**							
10	**30**	**17**	**10**	**6**	**4**	**4**	**1**						
11	**47**	**20**	**11**	**8**	**5**	**4**	**4**	**1**					
12	**57**	29	**15**	**11**	**6**	**4**	**4**	**4**	**1**				
13	**78**	34	21	**13**	**10**	**6**	**4**	**4**	**4**	**1**			
14	**91**	43	25	**15**	**11**	**8**	**5**	**4**	**4**	**4**	**1**		
15	**124**	56	31	**15**	**13**	**10**	**7**	**5**	**4**	**4**	**4**	**1**	
16	**140**	65	38	24	**14**	**11**	**8**	**6**	**4**	**4**	**4**	**4**	**1**
17	**183**	**68**	44	27	18	**13**	**11**	**7**	**6**	**4**	**4**	**4**	**4**
18	**207**	**94**	**48**	33	**21**	16	12	10	**6**	**5**	**4**	**4**	**4**
19	258	108	62	35	27	17	14	11	**9**	**6**	**5**	**4**	**4**
20	**285**	133	71	45	**28**	21	15	12	10	**8**	**6**	**4**	**4**
21	**352**	151	77	49	35	24	18	14	**11**	**9**	**7**	**5**	**4**
22	**385**	172	**77**	59	38	29	19	15	**11**	11	**8**	**6**	**5**
23	**466**	187	**104**	67	40	32	24	**15**	14	**11**	10	**7**	**6**
24	**510**	231	116	78	50	35	**24**	20	**14**	13	11	**8**	**6**
25	**600**	256	130	83	57	38	30	23	17	14	12	10	**8**
26	**650**	**260**	**130**	94	65	39	33	26	18	15	13	11	**9**
27	**763**	**319**	**167**	105	74	**39**	36	27	22	**15**	14	**11**	11
28	**819**	362	188	124	79	56	36	32	24	19	15	13	11
29	**950**	418	221	134	91	59	42	33	27	21	**15**	**14**	12
30	**1020**	462	225	142	97	66	46	37	30	24	**15**	15	13
31	1165	517	273	153	105	74	48	39	32	26	21	**15**	**14**
32	**1240**	579	300	169	106	78	60	40	**32**	29	23	18	**14**

1.36 **Table** Upper bounds on $C(v,k,4)$ for $v \leq 32$ and $k \leq 16$. Values known to be exact are in **bold**.

$t = 4$

$v \backslash k$	5	6	7	8	9	10	11	12	13	14	15	16
5	**1**											
6	**5**	**1**										
7	**9**	**5**	**1**									
8	**20**	**7**	**5**	**1**								
9	**30**	**12**	**6**	**5**	**1**							
10	**51**	**20**	**10**	**5**	**5**	**1**						
11	**66**	**32**	**17**	**9**	**5**	**5**	**1**					
12	**113**	41	24	**12**	**8**	**5**	**5**	**1**				
13	157	66	30	**18**	**10**	**7**	**5**	**5**	**1**			
14	230	80	44	24	**16**	**9**	**6**	**5**	**5**	**1**		
15	295	117	57	30	20	14	**8**	**5**	**5**	**5**	**1**	
16	405	152	76	**30**	26	18	**12**	**7**	**5**	**5**	**5**	**1**
17	491	188	99	53	28	23	15	**10**	**7**	**5**	**5**	**5**
18	664	236	126	66	38	26	19	**12**	**9**	**6**	**5**	**5**
19	846	325	152	84	48	32	23	17	**11**	**9**	**6**	**5**
20	1083	386	202	93	63	36	28	20	16	**10**	**8**	**5**
21	1251	490	237	127	75	51	31	25	18	14	**9**	**7**
22	1573	580	252	157	97	54	38	28	22	17	12	**9**
23	**1771**	720	**253**	196	109	77	42	31	25	20	15	11
24	**2237**	784	**357**	234	122	89	59	31	28	23	18	12
25	2706	992	440	263	168	98	70	47	30	27	21	17
26	3222	1154	558	298	198	119	82	55	37	28	24	18
27	3775	**1170**	670	350	216	138	99	65	42	31	27	22
28	4501	**1489**	817	428	267	160	109	79	55	31	29	24
29	5229	1803	956	512	314	198	119	94	68	43	31	28
30	5956	2220	1102	560	366	231	144	102	74	50	31	29
31	6595	2627	1176	617	435	278	165	115	80	63	**31**	30
32	7703	3119	1440	620	479	323	184	132	101	67	52	**30**

1.37 Table Upper bounds on $C(v,k,5)$ for $v \leq 32$ and $k \leq 16$. Values known to be exact are in **bold**.

$t = 5$

$v\backslash k$	6	7	8	9	10	11	12	13	14	15	16
6	**1**										
7	**6**	**1**									
8	**12**	**6**	**1**								
9	**30**	**9**	**6**	**1**							
10	**50**	**20**	**8**	**6**	**1**						
11	100	34	**16**	**7**	**6**	**1**					
12	**132**	59	**26**	**12**	**6**	**6**	**1**				
13	**245**	78	42	**19**	**11**	**6**	**6**	**1**			
14	371	138	55	32	**14**	**10**	**6**	**6**	**1**		
15	579	189	89	42	27	**13**	**9**	**6**	**6**	**1**	
16	808	283	117	61	34	22	**12**	**8**	**6**	**6**	**1**
17	1213	405	178	79	48	30	**17**	**11**	**7**	**6**	**6**
18	1547	583	256	113	54	42	24	**15**	**9**	**6**	**6**
19	2175	706	356	149	83	49	37	21	**14**	**9**	**6**
20	2850	1003	492	220	108	65	42	33	18	**12**	**8**
21	3930	1320	603	271	145	79	56	38	28	16	**12**
22	4681	1701	723	378	190	110	64	48	34	22	14
23	6162	2044	757	489	263	131	85	56	44	30	20
24	**7084**	2710	**759**	615	297	204	86	67	49	38	24
25	**9321**	3163	**1116**	717	398	232	145	74	58	47	35
26	11952	4151	1452	830	514	273	175	103	66	54	41
27	15174	4680	2010	960	622	354	208	125	77	61	49
28	18369	**4680**	2551	1224	771	424	261	172	90	62	55
29	22870	**6169**	3180	1608	920	561	321	218	137	67	61
30	27136	7800	3998	2009	1123	644	379	255	162	102	62
31	32365	9953	4567	2418	1395	799	482	293	197	133	62
32	35882	12469	4820	2965	1649	1002	588	363	240	159	**62**

1.38 Theorem $C(v, v-1, t) = t + 1$ for all t.

1.39 Theorem (Turán) Suppose $q = \left\lfloor \frac{v}{v-t-1} \right\rfloor$. Then $C(v, v-2, t) = qv - \binom{q+1}{2}(v - t - 1)$.

1.5 Structure of Optimal Coverings

1.40 Definition Let $(X, \mathcal{B})$ be a 2-$(v,k,1)$ covering. The *excess graph* of $(X, \mathcal{B})$ is the multigraph (X, E), where each edge xy occurs with multiplicity $|\{B \in \mathcal{B} : \{x,y\} \subseteq B\}| - 1$.

1.41 Table Optimal 2-$(v,3,1)$ coverings.

$v \equiv$	$C(v,3,2)$	Excess Graph	Construction
$1, 3 \pmod 6$	$\frac{v^2 - v}{6}$	Empty	$(v, 3, 1)$ BIBD
$0 \pmod 6$	$\frac{v^2}{6}$	$\frac{v}{2}K_2$	For $v \geq 18$, fill in each group of a $\{3\}$-GDD of type $6^{v/6}$ with an optimal covering on six points.
$2, 4 \pmod 6$	$\frac{v^2 + 2}{6}$	$K_{1,3} \cup \frac{v-4}{2}K_2$	For $v \equiv 4 \pmod 6$, $v \geq 22$, fill in each group of a $\{3\}$-GDD of type $6^{(v-4)/6}4^1$ with an optimal covering on four or six points; for $v \equiv 2 \pmod 6$, $v \geq 26$, fill in each group of a $\{3\}$-GDD of type $6^{(v-8)/6}8^1$ with an optimal covering on six or eight points.
$5 \pmod 6$	$\frac{v^2 - v + 4}{6}$	One edge of multiplicity 2	For $v \geq 11$, take a $(v, \{3, 5^*\})$-PBD on v points and fill in the block of size 5 with an optimal covering on five points.

1.42 Table Optimal 2-$(v, 4, 1)$ coverings, $v \notin \{7, 9, 10, 19\}$.

$v \equiv$	$C(v,4,2)$	Excess Graph	Construction
$1, 4 \pmod{12}$	$\frac{v^2 - v}{12}$	Empty	$(v, 4, 1)$ BIBD
$0, 6 \pmod{12}$	$\frac{v^2}{12}$	$\frac{v}{2}K_2$	For $v \geq 30$, fill in each group of a $\{4\}$-GDD of type $6^{v/6}$ with an optimal covering on six points.
$3, 9 \pmod{12}$	$\frac{v^2 + 3}{12}$	$K_{1,4} \cup \frac{v-5}{2}K_2$	For $v \geq 51$, fill in each group of a $\{4\}$-GDD of type $6^{(v-15)/6}15^1$ with an optimal covering on six or fifteen points.
$7, 10 \pmod{12}$	$\frac{v^2 - v + 6}{12}$	One edge of multiplicity three	Take a $(v, \{4, 22^*\})$-PBD and replace the block of size 22 by an optimal covering on 22 points.
$8, 11 \pmod{12}$	$\frac{v^2 + v}{12}$	A 2-regular multigraph on v points	Take an optimal covering on $v-1$ points in which the pair 12 occurs four times, and in which $\{1, 2, 3, 4\}$ is a block. Then replace the block $\{1, 2, 3, 4\}$ by the two blocks $\{1, 3, 4, v\}$ and $\{2, 3, 4, v\}$. Finally, adjoin new blocks $\{5, 6, 7, v\}, \ldots, \{v-3, v-2, v-1, v\}$.
$2, 5 \pmod{12}$	$\frac{v^2 + v + 6}{12}$	A multigraph on v points in which two vertices have degree 5 and the remaining $v-2$ vertices have degree 2; or one in which one vertex has degree 8 and the remaining $v-1$ vertices have degree 2	Take a $(v-1, 4, 1)$ BIBD and adjoin new blocks $\{1, 2, 3, v\}$, $\{4, 5, 6, v\}$, $\ldots$, $\{v-4, v-3, v-2, v\}$ and $\{v-3, v-2, v-1, v\}$.

1.6 Resolvable Coverings with $\lambda = 1$

1.43 Definition A t-(v, k, λ) covering $(X, \mathcal{B})$ is *resolvable* if $\mathcal{B}$ can be partitioned into *parallel classes*, each of which consists of v/k disjoint blocks.

1.44 Example A resolvable 2-$(24, 4, 1)$ covering on $48 = L(24, 4, 2)$ blocks [6]. Let $X = \mathbb{Z}_3 \times \{0, \ldots, 7\}$. Three parallel classes are formed by developing the following class modulo 3:

$$\{(0,0),(1,0),(1,1),(0,2)\}\ \{(2,0),(2,3),(0,4),(0,6)\}\ \{(0,1),(1,2),(0,3),(1,3)\}$$
$$\{(2,1),(2,2),(0,5),(0,7)\}\ \{(1,4),(1,5),(2,6),(2,7)\}\ \{(2,4),(2,5),(1,6),(1,7)\}.$$

Three more parallel classes are formed by developing the following class modulo 3:

$$\{(0,0),(2,1),(0,4),(2,4)\}\ \{(1,0),(2,3),(1,5),(2,5)\}\ \{(2,0),(1,3),(1,7),(2,7)\}$$
$$\{(0,1),(1,1),(1,6),(0,7)\}\ \{(0,2),(1,2),(1,4),(0,5)\}\ \{(2,2),(0,3),(0,6),(2,6)\}.$$

The seventh parallel class is formed by developing the following two blocks modulo 3:

$$\{(0,0),(0,1),(2,5),(2,6)\} \ \{(0,2),(0,3),(2,4),(2,7)\}.$$

Finally, the eighth parallel class is formed by developing the following two blocks modulo 3:

$$\{(0,0),(1,2),(0,6),(1,7)\} \ \{(0,1),(2,3),(2,4),(0,5)\}.$$

1.45 Theorem Suppose $v \equiv 0 \pmod{k}$. If $v-1 \equiv 0 \pmod{(k-1)}$, then a resolvable 2-$(v,k,1)$ covering with $L(v,k,2)$ blocks is equivalent to a resolvable $(v,k,1)$ BIBD.

1.46 Theorem There exists a resolvable 2-$(v,3,1)$ covering having $L(v,3,2)$ blocks for all $v \equiv 0 \pmod{6}$, $v \geq 18$.

1.47 Theorem There exists a resolvable 2-$(v,4,1)$ covering having $L(v,4,2)$ blocks for all $v \equiv 0 \pmod{4}$, $v \neq 12$, except possibly when $v \in \{108,116,132,156,204,212\}$.

1.48 Remark Theorem 1.47 is a recent result of Abel, Assaf, Bennett, Bluskov, and Greig, eliminating four of the open cases from [6].

1.49 Definition Let $r(q,k)$ denote the minimum number of parallel classes in a resolvable $2-(kq,k,1)$ covering.

1.50 Theorem (Haemers) $r(q,k) \geq q+1$. Further, equality holds if and only if q divides k and q is the order of an affine plane.

1.51 Theorem (Haemers) Suppose that q is the order of an affine plane, and k is a positive integer such that $\lceil k/q \rceil \leq 2k/(2q-1)$. Then $r(q,k) \leq q+2$.

1.52 Theorem [8] The following values of $r(q,k)$ for small q are known:

1. $r(2,k) = \begin{cases} 3 & k \text{ even}, \\ 4 & k \text{ odd}. \end{cases}$
2. $r(3,k) = \begin{cases} 4 & k \equiv 0 \pmod{3}, \\ 5 & \text{otherwise}. \end{cases}$
3. $r(4,k) = \begin{cases} 5 & k \equiv 0 \pmod{4}, \\ 7 & k = 2,3, \\ 6 & \text{otherwise}. \end{cases}$

1.53 Definition A resolvable $2-(kq,k,1)$ covering is *equitable* if every pair of points occurs in either one or two blocks.

1.54 Theorem Let $s = (qk-1)/(k-1)$. If an equitable resolvable $2-(kq,k,1)$ covering with r parallel classes exists, then $s \leq r \leq 2s$.

1.55 Theorem If an equitable resolvable $2-(kq,k,1)$ covering exists, then

$$k < 2q - \sqrt{2q - 9/4}.$$

1.7 See Also

§I.??	BIBDs are coverings with void excess graph.
§II.??	Incomplete transversal designs are used extensively in various constructions for coverings.
§III	Pairwise balanced designs and group divisible designs.
§IV.??	t-wise balanced designs.
§V.??	Gives the connection between coverings, Turán designs, and lottery schemes.
[7]	A general survey of coverings and packings with an extensive bibliography.
[4]	Up-to-date numerical results and tables of the best known coverings.
[8]	Information on resolvable coverings.
[5]	Computational methods for coverings.

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