New Constructions for Covering Designs

Daniel M. Gordon[*] Greg Kuperberg[†] Oren Patashnik[*]

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Abstract

A (v, k, t) *covering design*, or *covering*, is a family of k-subsets, called blocks, chosen from a v-set, such that each t-subset is contained in at least one of the blocks. The number of blocks is the covering's *size*, and the minimum size of such a covering is denoted by $C(v, k, t)$. This paper gives three new methods for constructing good coverings: a greedy algorithm similar to Conway and Sloane's algorithm for lexicographic codes [6], and two methods that synthesize new coverings from preexisting ones. Using these new methods, together with results in the literature, we build tables of upper bounds on $C(v, k, t)$ for $v \le 32$, $k \le 16$, and $t \le 8$.

1 Introduction

Let the covering number $C(v, k, t)$ denote the smallest number of k-subsets of a v-set that cover all t-subsets. These numbers have been studied extensively. Mills and Mullin [19] give known results and many references. Hundreds of papers have been written for particular values of v, k, and t. The best general lower bound on $C(v, k, t)$, due to Schönheim [27], comes from the following inequality:

Theorem 1

$$C(v,k,t) \;\ge\; \left\lceil \frac{v}{k}\, C(v-1,k-1,t-1) \right\rceil .$$

[*]Center for Communications Research, 4320 Westerra Ct., San Diego, CA 92121
[†]Department of Mathematics, University of Chicago, Chicago, IL 60637

Iterating this gives the Schönheim bound $C(v,k,t) \geq L(v,k,t)$, where

$$L(v,k,t) \;=\; \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \cdots \left\lceil \frac{v-t+1}{k-t+1} \right\rceil \cdots \right\rceil \right\rceil .$$

Sometimes a lower bound of de Caen [7] is slightly better than the Schönheim bound when k and t are not too small:

$$C(v,k,t) \;\geq\; \frac{(t+1)(v-t)}{(k+1)(v-k)} \binom{v}{t} \Big/ \binom{k}{t} .$$

The best general upper bound on $C(v,k,t)$ is due to Rödl [26]: Define the *density* of a covering to be the average number of blocks containing a t-set. The minimum density of a (v,k,t) covering is $C(v,k,t)\binom{k}{t}/\binom{v}{t}$ and is obviously at least 1. Rödl shows that for k and t fixed there exist coverings with density approaching 1 as v gets large. Erdős and Spencer [11] give the bound

$$C(v,k,t)\binom{k}{t}\Big/\binom{v}{t} \;\leq\; 1 + \ln\binom{k}{t} ,$$

which is weaker but applies to all v, k, and t. Furthermore it can be improved by at most a factor of $4\ln 2 \approx 2.77$ asymptotically, because a $(v, v-1, \lfloor v/2 \rfloor)$ covering that achieves the Schönheim lower bound has density asymptotic to $v/4$, while the Erdős-Spencer upper bound in that case corresponds to a density asymptotic to $v \ln 2$.

This paper presents new constructions for coverings. The greedy method of Section 2 produces reasonably good coverings and it is completely general—it applies to all possible values of v, k, and t, and it doesn't rely on the existence of other good coverings. The finite geometries of Section 3 produce very good (often optimal) coverings, but they apply only to certain sets of v, k, and t values. The induced-covering method of Section 4, which constructs coverings from larger ones, and the dynamic programming method of Section 5, which constructs coverings from smaller ones, both apply to all parameter values, but they rely on preexisting coverings. (We show in a paper with Spencer [12] that the greedy construction, as well as the induced-covering method applied to certain finite geometry coverings, both produce coverings that match Rödl's bound.) Finally, the previously known methods of Section 6, when combined with the methods of earlier sections, yield the tables of upper bounds in Section 7.

2 Greedy Coverings

Our greedy algorithm for generating coverings is analogous to the surprisingly good greedy algorithm of Conway and Sloane [6] for generating codes. That algorithm may be stated very concisely: To construct a code of length n and minimum distance d, arrange the binary n-tuples in lexicographic order, and repeatedly choose the first one in the list that is distance d or more from all n-tuples chosen earlier; the n-tuples chosen are the codewords. The resulting code is called a *lexicographic code*, or *lexicode*.

This simple method has several nice features: Lexicodes tend to be fairly good (at packing codewords into the space), they are linear, and they include some well-known codes such as Hamming codes and the binary Golay codes. Brouwer, Shearer, Sloane, and Smith [3, page 1349] use the same method to make constant weight codes, by choosing only n-tuples of a given weight.

The greedy algorithm does not require lexicographic order. Brualdi and Pless [4] show that a large family of orders lead to linear codes. And sometimes Gray code orders, for example, lead to better codes.

Constructing good codes and good constant weight codes are packing problems. But a similar method applies to covering problems. A greedy (v, k, t) covering is one generated by the following algorithm:

1. Arrange the k-subsets of a v-set in a list.
2. Choose from the list the k-subset that contains the maximum number of t-sets that are still uncovered. In case of ties, choose the k-subset occurring earliest in the list.
3. Repeat Step 2 until all t-sets are covered.

The list of k-sets can be in any order. Some natural orders are lexicographic, colex (which is similar to lexicographic but the subsets are read from right to left rather than left to right), and a generalized Gray code order (where successive sets differ only by one deletion and one addition). The resulting lists, when $k = 3$ and $v = 5$, are

123	124	125	134	135	145	234	235	245	345	(lexicographic);
123	124	134	234	125	135	235	145	245	345	(colex);
123	134	234	124	145	245	345	135	235	125	(gray).

Nijenhuis and Wilf [22] give algorithms to generate lexicographic and Gray code orders. Stanton and White [30] discuss colex algorithms.

It is natural to investigate the greedy algorithm with random order, too, since we know [12] that random order does well asymptotically. To keep with the constructive spirit of this paper, we used an easily reproduced "random" permutation of the k-sets. To generate the permutation, start with the k-sets lexicographically ordered in positions 1 through $\binom{v}{k}$, then successively swap the k-sets in positions i and $i+j$, for $i = 1, 2, \ldots, \binom{v}{k}$, where j is $X_i \bmod (\binom{v}{k} - i + 1)$ and where the sequence of pseudo-random X's comes from the linear congruential generator $X_{i+1} = (41X_i + 7) \bmod 2^{30}$. The seed X_0 is 1, and when there are multiple random-order runs on the same set of (v, k, t) parameters, the subsequent seeds are 2, 3, ... Knuth [15] discusses the linear congruential method.

Greedy coverings are not in general optimal, but as happens with codes (Brouwer, Shearer, Sloane, and Smith [3], Brualdi and Pless [4], Conway and Sloane [6]) they are often quite good—about 42% of the table entries come from greedy coverings. Interestingly, the Steiner system $S(24, 8, 5)$, which Conway and Sloane [6, page 347] showed is a constant-weight lexicographic code, also arises as a greedy covering.

The problem with greedy coverings is that they are expensive to compute. Our implementation of the algorithm above uses two arrays: one with $\binom{v}{k}$ locations corresponding to the k-subsets, and one with $\binom{v}{t}$ locations corresponding to the t-subsets. Each k-set array location contains the number of uncovered t-sets contained in that k-set, and is initialized to $\binom{k}{t}$. Each t-set array location contains a 0 or 1, indicating whether that t-set has been covered. Each time through Step 2, each t-set contained in the selected k-set must be checked. If the t-set is uncovered, it is marked as covered, and each k-set containing it must have its array location decremented. For fixed k and t, the algorithm asymptotically takes time and space $O(v^k)$.

We ran a program to generate greedy coverings for all entries in our tables, for all four orders described above. For random order, we used 10^e runs, where $e = 3[v \le 20] + [v \le 15] + [v \le 10] + [k \le 10] + [k \le 5] + 2[U]$ and where U is the predicate '$t = 2$ and $C(v, k, 2)$ is unknown' (the symbol $[P]$ is 1 if the predicate P is true, 0 otherwise).

For the range of parameters of our tables, the four orders produced coverings of roughly the same size, but lexicographic order performed slightly better on average than colex order, which performed better than Gray code order, which performed better than a single run of random order.

3 Finite Geometry Coverings

Finite geometries may be used to construct very good coverings for certain sets of parameters. Anderson [2] has a nice discussion of finite geometries.

Let $\mathrm{PG}(m,q)$ denote the projective geometry of dimension m over $\mathrm{GF}(q)$, where q is a prime power. The points of $\mathrm{PG}(m,q)$ are the equivalence classes of nonzero vectors $u = (u_0, u_1, \ldots, u_m)$, where two vectors u and v are equivalent if $u = \lambda v$ for some nonzero $\lambda \in \mathrm{GF}(q)$. There are $(q^{m+1}-1)/(q-1)$ such points.

A k-flat is a k-dimensional subspace of $\mathrm{PG}(m,q)$, for $1 \le k \le m$, determined by $m-k$ independent homogeneous linear equations. A k-flat has $(q^{k+1}-1)/(q-1)$ points, and there are $\begin{bmatrix} m+1 \\ k+1 \end{bmatrix}_q$ different k-flats in $\mathrm{PG}(m,q)$, where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\ldots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\ldots(q-1)}$$

is the q-binomial coefficient.

By removing all points with $u_0 = 0$ we obtain the affine (or Euclidean) geometry $\mathrm{AG}(m,q)$. It has q^m points and $q^{m-k}\begin{bmatrix} m \\ k \end{bmatrix}_q$ different k-flats, each of which contains q^k points.

For either geometry, any $k+1$ independent points determine a k-flat, and $k+1$ dependent points are contained in multiple k-flats, so the k-flats cover every set of $k+1$ points. Thus, taking the points of the geometry as the v-set of the covering, and taking the points of a k-flat as a block of the covering, we get the following two theorems.

Theorem 2

$$C\left(\frac{q^{m+1}-1}{q-1}, \frac{q^{k+1}-1}{q-1}, k+1\right) \le \begin{bmatrix} m+1 \\ k+1 \end{bmatrix}_q .$$

Theorem 3

$$C(q^m, q^k, k+1) \le q^{m-k}\begin{bmatrix} m \\ k \end{bmatrix}_q .$$

Equality holds for both theorems when $k = m-1$ or $k = 1$. Theorem 2 is due to Ray-Chaudhuri [25], and Theorem 3 follows easily from results of Abraham, Ghosh, and Ray-Chaudhuri [1], although the idea of using finite geometries to construct coverings dates back at least to Veblen and Bussey [38] in 1906.

4 Induced coverings

The main drawback of the finite geometry coverings is that they exist only for certain families of parameters. But they are such good coverings that they can be used to construct pretty good coverings for other parameters.

Suppose we have a good (v, k, t) covering, say from a geometry, and we want to construct a (v', k', t) covering, where $v' < v$ and $k' < k$. Consider the family of sets obtained from the (k-element) blocks by randomly choosing v' elements of the v-set, deleting all other elements from the blocks, and throwing out any blocks with fewer than t elements (since those blocks cover no t-sets).

The remaining blocks cover all t-subsets of the v' elements, but have different sizes. Suppose some block has ℓ elements. If $\ell = k'$ its size is correct as is, and it becomes a block of our new covering. If $\ell < k'$, add any $k' - \ell$ elements to the block. And if $\ell > k'$, replace the block by an (ℓ, k', t) covering, which covers all t-sets the original block covered.

The new blocks each have k' elements, and together they cover all t-sets, so the new family forms a (v', k', t) *induced* covering.

In small cases, the method tends to do best when k'/k is about v'/v. In large cases, the method does well if for every ℓ near $v'k/v$, a good (ℓ, k', t) covering is available. Also, it need not start with a finite geometry covering—any (v, k, t) covering will do. But generally the better the covering it starts with, the better the result.

The induced coverings in our tables come either from using the simple special cases of Section 6.1 or from finite geometries. We constructed each finite geometry covering based on $PG(m, p)$ and $AG(m, p)$ with $p \leq 11$ prime and with at most 10^4 points and 10^6 flats. For each such covering, and for each v and k in the relevant table, we used a random set of v points to construct an induced covering as described above, trying 100 random sets in each case.

5 Combining Smaller Coverings

Suppose we want to form a (v_1+v_2, k, t) covering. Let the (v_1+v_2)-set be the disjoint union of a v_1-set and a v_2-set. Given an s with $0 \leq s \leq t$, choose a (v_1, ℓ, s) covering and a $(v_2, k-\ell, t-s)$ covering for some ℓ, which must be in the range $s \leq \ell \leq k - t + s$. For each possible arrangement of t elements as an s-subset of the v_1-set and a $(t-s)$-subset of the v_2-set, there is an ℓ-set from the first covering and a $(k-\ell)$-set from the second covering whose union is a k-set that covers the t-set. Thus the number of blocks that cover all such t-sets is at most the product of the sizes of the two coverings. Choosing an optimal ℓ for each s gives us our (v_1+v_2, k, t) covering built up from smaller coverings. This construction gives the bound

$$C(v_1+v_2, k, t) \;\leq\; \sum_{s=0}^{t} \min_{\ell} C(v_1, \ell, s) \cdot C(v_2, k-\ell, t-s)\,.$$

Furthermore we can try all choices of v_1 and v_2 summing to the v of interest.

The coverings produced by this method tend to have some redundancy. To remove redundancy when $v_1 = 2$, for example, we can try combining a (v, k, t) covering and a $(2, 0, 0)$ covering (which has one block, the empty set), along with a $(v, k-2, t-1)$ covering and a $(2, 2, 2)$ covering. This forms a $(v+2, k, t)$ covering, and is sometimes an improvement over the basic construction above:

$$C(v+2, k, t) \;\leq\; C(v, k, t) \;+\; C(v, k-2, t-1)\,.$$

This example has replaced the s and $s+1$ terms of the basic construction's bound, when $s = 1$, with the single term

$$\min_{\ell} C(v_1, \ell, s+1) \cdot C(v_2, k-\ell, t-s)\,.$$

The new term corresponds to covering any t-subset having either s or $s+1$ elements in the v_1-set, by using one product of coverings, rather than two. If changing $C(v_1, \ell, s)$ to $C(v_1, \ell, s+1)$ does not cost too much, the bound will improve.

To generalize this combining of terms, define $c_{i,j}$ for $0 \leq i \leq j \leq t$ to be the number of blocks required to cover any t-subset that has between i and j elements in the v_1-set, and between $t-j$ and $t-i$ elements in the v_2-set. Since $c_{i,j} \leq c_{i,r} + c_{r+1,j}$ for any $i \leq r < j$, we have

$$c_{i,j} \;\leq\; \min\Bigl(\min_{\ell} C(v_1, \ell, j) \cdot C(v_2, k-\ell, t-i),\; \min_{i \leq r < j} (c_{i,r} + c_{r+1,j})\Bigr)\,.$$

Using dynamic programming, we may efficiently compute a bound for $c_{0,t}$, which is an upper bound for $C(v_1+v_2, k, t)$.

This general construction produces about 30% of the entries in our tables. It includes as special cases several of the simple constructions of Section 6.1, as well as the direct-product construction of Morley and van Rees [21], which yields the bound

$$C(2v+y, v+k+y, t+s+1) \;\le\; C(v,k,t) \;+\; C(v+y, k+y, s)\,.$$

6 Other Constructions

6.1 Simple Constructions

There are several simple and well-known methods for building coverings from other coverings. All but the last of these methods are special cases of the methods in the previous two sections.

Adding a random element to each block of a (v,k,t) covering gives a $(v,k+1,t)$ covering of the same size. Thus

$$C(v,k+1,t) \;\le\; C(v,k,t)\,.$$

Adding a new element to a v-set, and including it in every block in a (v,k,t) covering, forms a $(v+1,k+1,t)$ covering of the same size, hence

$$C(v+1,k+1,t) \;\le\; C(v,k,t)\,.$$

Combining a (v,k,t) covering and a $(v,k-1,t-1)$ covering over the same v-set, by adding a new v+1st element to all of the blocks of the $(v,k-1,t-1)$ covering but to none of the blocks of the (v,k,t) covering, forms a $(v+1,k,t)$ covering, of size the sum of the other two sizes, thus

$$C(v+1,k,t) \;\le\; C(v,k,t) \;+\; C(v,k-1,t-1)\,.$$

Those constructions are special cases of the method of Section 5.

Deleting one element from a v-set, and adding a random element to any block of a (v,k,t) covering that contains the deleted element, creates a $(v-1,k,t)$ covering of the same size. Thus

$$C(v-1,k,t) \;\le\; C(v,k,t)\,.$$

Choosing the element of a covering that occurs in the fewest blocks, throwing away all other blocks, and then throwing away the chosen element, results in a $(v-1, k-1, t-1)$ covering. This method, due to Schönheim, is a reformulation of Theorem 1; the corresponding upper bound is

$$C(v-1, k-1, t-1) \ \leq \ \left\lfloor \frac{k}{v} \, C(v,k,t) \right\rfloor .$$

Those two constructions are special cases of the induced-covering method of Section 4.

Replacing each element of the v-set in a (v, k, t) covering by m different elements gives an (mv, mk, t) covering of the same size, thus

$$C(mv, mk, t) \ \leq \ C(v, k, t) \, .$$

6.2 Steiner Systems

A *Steiner system* is a covering in which the covering density is 1—every t-set is covered exactly once. Clearly a Steiner system is an optimal covering, as well as an optimal packing, and $C(v, k, t) = L(v, k, t)$. The projective and affine coverings by lines (1-flats), for example, are Steiner systems. Brouwer, Shearer, Sloane, and Smith [3, page 1342] and Chee, Colbourn, and Kreher [5] give tables of small Steiner systems.

If a (v, k, t) Steiner system exists then $C(v+1, k, t) = L(v+1, k, t)$. This result is due to Schönheim [27, Theorem II]; the proof also appears in Mills and Mullin [19, Theorem 1.3].

6.3 Turán Theory

The *Turán number* $T(n, \ell, r)$ is the minimum number of r-subsets of an n-set such that every ℓ-subset contains at least one of the r-subsets. It is easy to see that

$$C(v, k, t) \ = \ T(v, v-t, v-k) \, ,$$

so covering numbers are just Turán numbers reordered. The two sets of numbers, however, have been studied for different parameter ranges (de Caen's lower bound in the introduction, for instance, is useful primarily for Turán theory ranges). Most papers on coverings have v large compared with k and t, while most papers on Turán numbers have n large compared with ℓ and r, often focusing on the quantity

$\lim_{n\to\infty} T(n,\ell,r)/\binom{n}{r}$ for fixed ℓ and r. Thus Turán theory usually studies $C(v,k,t)$ for k and t not too far from v.

Fifty years ago Turán [37] determined $T(n,\ell,2)$ exactly, showing that $C(v,v-2,t) = L(v,v-2,t)$, the Schönheim lower bound. He also gave upper bounds and conjectures for $T(n,4,3)$ and $T(n,5,3)$, which stimulated much of the research. The results labeled 'Turán theory' in our tables either are described in recent survey papers by de Caen [8] and Sidorenko [29], or follow from constructions due to de Caen, Kreher, and Wiseman [10] or to Sidorenko [28].

Sidorenko [28] also recently told us of a Turán theory construction, similar in spirit to the combining constructions of Section 5, that improves many bounds in the table. In terms of covering theory, let x be an element occurring in the most blocks of a (v,k,t) covering, and replace x by x' and x'': If a block b did not contain x, replace it by two blocks, $b\cup\{x'\}$ and $b\cup\{x''\}$; if b did contain x, replace it by the single block $b-\{x\}\cup\{x',x''\}$. Finally, add a $(v-1,k+1,t+1)$ covering on the same elements minus x' and x''. It is not hard to see that this is a $(v+1,k+1,t+1)$ covering, and that it gives the bound

$$C(v+1,k+1,t+1) \;\le\; \lfloor (2v-k)\,C(v,k,t)/v\rfloor \;+\; C(v-1,k+1,t+1)\,.$$

6.4 Cyclic Coverings

Another well-known method that is often successful when applicable—when the size of a prospective covering is v—is to construct a cyclic covering: Choose some k-subset as the first block, and choose the $v-1$ cyclic shifts of that block as the remaining blocks. Trying this for all possible k-sets is fairly cheap, and frequently it produces a covering. The entries $C(19,9,3) \le 19$ and $C(24,10,3) = 24$ in our tables, for example, are generated by the k-sets 1 2 3 4 6 8 13 14 17 and 1 2 3 5 6 8 12 13 15 21, and are unmatched by any other method.

Incidentally, if the size of a prospective covering is a multiple of v, say $2v$, the same method applies by taking the cyclic shifts of two starting blocks; the few cases we tried for this variation produced no improvements in the tables.

6.5 Hill-Climbing

For cases of interest—with v not too large—random coverings are not very good, but hill-climbing sometimes finds good coverings: Start with a fixed number of random k-sets, say $L(v,k,t)+\epsilon$ for some small integer ϵ. Rank the k-sets by the number of t-sets they cover that no other k-set covers, and replace one with lowest rank by another random k-set. Repeat until all t-sets are covered or until time runs out.

We found a few good coverings with this method, but Nurmela and Östergård [23] went much further, using simulated annealing—a more sophisticated hill-climbing—to find many good coverings. In fact many of the bounds in the tables could be improved, by starting with a covering produced by one of the other methods and then hill-climbing; but generally the improvements would be small.

7 Tables of Upper Bounds on $C(v,k,t)$

We constructed Tables 2 through 8 using the methods described above, together with results from the literature. Each table entry indicates the upper bound, the method of construction, and whether the covering is known to be optimal. We have tried to provide constructions for as many sets of parameters as possible, so we list a method of construction from this paper even when a result in the literature achieves the same bound. When two different methods produce the same size covering, we've given precedence to the method listed earlier in the Key to the tables.

About 93% of the 1631 nontrivial ($v>k>t$) upper bounds in the tables come from one of the constructions described in this paper. For each of the remaining upper bounds, there is a source in our reference list that describes the result, although to keep our reference list reasonably short we have often given a secondary source rather than the original. (Mills and Mullin [19] give an extensive list of previous results and references.) Sources for Steiner systems, Turán number bounds, and simulated annealing coverings appear in Sections 6.2, 6.3, and 6.5; the Todorov constructions come from papers by Todorov [31, 33, 34] and Todorov and Tonchev [36]; and the remaining upper bounds appear in Table 1. The covering number $C(24,18,17)$ is listed in Table 1, even though it doesn't occur in the other tables, because it yields a $(15,9,8)$ simple induced covering (of Section 6.1).

Gordon et al. [13] construct an optimal $(12,6,3)$ covering, using a block-array construction. That method directly extends to the

bound	reference
$C(29,5,2) \leq 44$	Lamken [16]
$C(31,7,2) = 26$	Todorov [34] techniques (lower bound)
$C(12,6,3) = 15$	Gordon et al. [13]
$C(14,6,3) \leq 25$	Lotto covering [17]
$C(15,6,3) \leq 31$	Lotto covering [17]
$C(16,6,3) \leq 38$	Hoehn [14]
$C(18,6,3) = 48$	Lotto covering [17]
$C(30,6,3) \leq 237$	Lotto covering [17]
$C(11,7,4) = 17$	Sidorenko [28]
$C(14,6,4) \leq 87$	Hoehn [14]
$C(18,6,4) \leq 258$	Lotto covering [17]
$C(18,9,4) \leq 43$	Gordon et al. [13]
$C(20,10,4) \leq 43$	block-array construction
$C(24,12,5) \leq 86$	block-array construction
$C(30,15,5) \leq 120$	block-array construction
$C(12,8,6) \leq 51$	Morley [20]
$C(32,16,6) \leq 286$	block-array construction
$C(15,12,8) = 30$	Radziszowski and Sidorenko [24]
$C(24,18,17) = 21252$	de Caen [8]

Table 1: Miscellaneous results

$(18,9,4)$ covering given in Table 1, and a similar construction gives four other coverings listed in the table.

Most of the lower bounds used to establish optimality follow from the Schönheim inequality (Theorem 1); and a few others are listed as equalities in Table 1. For the rest: If $t = 2$, the lower bound is explained by Mills and Mullin [19] when it is less than 14 or has $v \leq 5$, or explained by Todorov [34] otherwise; if $t = 3$, it's either Mills and Mullin or Todorov and Tonchev [36]; and if $4 \leq t \leq 8$, it's either Mills [18, Theorem 2.3], Todorov [32, Theorem 4], or Sidorenko's Turán theory survey [29].

How good are our bounds? For $t = 2$, very good—most of the entries are known to be optimal, and the largest gap between an entry's lower and upper bound is currently only a factor of 1.12. That largest gap rises with t, though, to 1.89 for $t = 4$, to 2.98 for $t = 6$, and to

3.72 for $t = 8$. We believe that our lower bounds tend to be closer to the truth than our upper bounds; it's quite possible that all the upper bounds are within a factor of 3, but probably not a factor of 2, of optimal.

Most of the entries in the tables for $t > 2$ are not optimal, and we would appreciate knowing of any better coverings. Please send communications to the first author, at `gordon@ccrwest.org`.

Key to Tables 2 through 8

l — greedy covering, lexicographic order
c — greedy covering, colex order
g — greedy covering, Gray code order
r — greedy covering, random order
p — projective geometry covering
a — affine geometry covering
o — cyclic covering
m — multiple of smaller covering
e — simple dynamic programming (Section 6.1)
j — simple induced covering (Section 6.1)
d — dynamic programming method (Section 5)
i — induced covering
u — Sidorenko Turán construction (Section 6.3)
s — Steiner system
t — Turán theory
x — covering with small k and t; see Mills and Mullin [19, §3]
y — covering with fixed size; see Mills and Mullin [19, §4]
v — Todorov construction
w — was known previously; see Table 1
n — Nurmela-Östergård simulated annealing covering
h — hill-climbing
$*$ — optimal covering

$v \backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	1^{*}													
4	3^{l*}	1^{*}												
5	4^{l*}	3^{l*}	1^{*}											
6	6^{o*}	3^{l*}	3^{l*}	1^{*}										
7	7^{l*}	5^{l*}	3^{l*}	3^{l*}	1^{*}									
8	11^{l*}	6^{l*}	4^{l*}	3^{l*}	3^{l*}	1^{*}								
9	12^{r*}	8^{l*}	5^{l*}	3^{l*}	3^{l*}	3^{l*}	1^{*}							
10	17^{r*}	9^{l*}	6^{j*}	4^{m*}	3^{l*}	3^{l*}	3^{l*}	1^{*}						
11	19^{r*}	11^{o*}	7^{r*}	6^{l*}	4^{l*}	3^{l*}	3^{l*}	3^{l*}	1^{*}					
12	24^{r*}	12^{o*}	9^{r*}	6^{l*}	5^{l*}	3^{l*}	3^{l*}	3^{l*}	3^{l*}	1^{*}				
13	26^{r*}	13^{l*}	10^{l*}	7^{l*}	6^{l*}	4^{d*}	3^{l*}	3^{l*}	3^{l*}	3^{l*}	1^{*}			
14	33^{r*}	18^{l*}	12^{l*}	7^{m*}	6^{y*}	5^{l*}	4^{l*}	3^{l*}	3^{l*}	3^{l*}	3^{l*}	1^{*}		
15	35^{l*}	19^{r*}	13^{r*}	10^{l*}	7^{l*}	6^{l*}	4^{m*}	3^{l*}	3^{l*}	3^{l*}	3^{l*}	3^{l*}	1^{*}	
16	43^{l*}	20^{a*}	15^{r*}	10^{l*}	8^{y*}	6^{l*}	5^{l*}	4^{m*}	3^{l*}	3^{l*}	3^{l*}	3^{l*}	3^{l*}	1^{*}
17	46^{r*}	26^{c*}	16^{r*}	12^{l*}	9^{r*}	7^{l*}	6^{l*}	5^{l*}	4^{l*}	3^{l*}	3^{l*}	3^{l*}	3^{l*}	3^{l*}
18	54^{r*}	27^{x*}	18^{o*}	12^{m*}	10^{y*}	7^{y*}	6^{m*}	5^{m*}	4^{d*}	3^{l*}	3^{l*}	3^{l*}	3^{l*}	3^{l*}
19	57^{j*}	31^{x*}	19^{o*}	15^{r}	11^{l*}	9^{l*}	7^{l*}	6^{l*}	5^{l*}	4^{e*}	3^{l*}	3^{l*}	3^{l*}	3^{l*}
20	67^{r*}	35^{r*}	21^{c*}	16^{v*}	12^{l*}	9^{r*}	7^{j*}	6^{l*}	6^{l*}	4^{m*}	4^{l*}	3^{l*}	3^{l*}	3^{l*}
21	70^{j*}	37^{x*}	21^{l*}	17^{v*}	13^{l*}	11^{l*}	7^{m*}	7^{l*}	6^{l*}	5^{l*}	4^{e*}	3^{l*}	3^{l*}	3^{l*}
22	81^{r*}	39^{x*}	27^{l*}	19^{m*}	13^{y*}	11^{l*}	9^{y*}	7^{m*}	6^{y*}	6^{l*}	5^{l*}	4^{m*}	3^{l*}	3^{l*}
23	85^{j*}	46^{x*}	28^{l*}	21^{v}	16^{v}	12^{l*}	10^{l*}	8^{j*}	7^{l*}	6^{l*}	5^{l*}	4^{d*}	4^{l*}	3^{l*}
24	96^{j*}	48^{x*}	30^{j*}	22^{v}	17^{v*}	12^{m*}	11^{l*}	8^{y*}	7^{j*}	6^{l*}	6^{l*}	5^{l*}	4^{m*}	3^{l*}
25	100^{j*}	50^{j*}	30^{a*}	23^{v*}	18^{v*}	13^{j*}	11^{l*}	10^{l*}	7^{y*}	7^{l*}	6^{l*}	5^{y*}	4^{m*}	4^{e*}
26	113^{e*}	59^{e*}	37^{e*}	24^{v*}	20^{j}	13^{m*}	12^{l*}	10^{m*}	8^{y*}	7^{m*}	6^{y*}	6^{l*}	5^{l*}	4^{m*}
27	117^{a*}	61^{x*}	38^{x*}	27^{o*}	20^{v*}	17^{l}	12^{m*}	11^{l*}	9^{y*}	7^{j*}	7^{l*}	6^{l*}	5^{m*}	5^{l*}
28	131^{e*}	63^{s*}	43^{d}	28^{o*}	22^{v}	18^{r}	14^{j}	11^{l*}	10^{l*}	7^{m*}	7^{e*}	6^{l*}	6^{l*}	5^{l*}
29	136^{j*}	73^{e*}	44^{w}	31^{j}	24^{v}	18^{l*}	14^{v*}	12^{l*}	10^{y*}	9^{j*}	7^{e*}	7^{l*}	6^{l*}	6^{l*}
30	150^{j*}	75^{x*}	48^{x*}	31^{j*}	25^{v}	19^{m*}	15^{v*}	13^{m*}	11^{y*}	9^{m*}	8^{j*}	7^{m*}	6^{m*}	6^{l*}
31	155^{l*}	78^{x*}	50^{x*}	31^{p*}	26^{v*}	20^{j*}	18^{l}	13^{y*}	12^{l*}	10^{l*}	8^{y*}	7^{j*}	7^{l*}	6^{l*}
32	171^{l*}	88^{x*}	54^{j}	38^{e*}	31^{l}	20^{m*}	19^{r}	15^{m}	12^{l*}	10^{m*}	9^{y*}	7^{y*}	7^{e*}	6^{l*}

Table 2: $t = 2$

$v \backslash k$	4	5	6	7	8	9	10	11	12	13	14	15	16
4	1^{*}												
5	4^{l*}	1^{*}											
6	6^{o*}	4^{l*}	1^{*}										
7	12^{r*}	5^{l*}	4^{l*}	1^{*}									
8	14^{l*}	8^{o*}	4^{l*}	4^{l*}	1^{*}								
9	25^{l*}	12^{l*}	7^{j*}	4^{l*}	4^{l*}	1^{*}							
10	30^{r*}	17^{r*}	10^{l*}	6^{l*}	4^{l*}	4^{l*}	1^{*}						
11	47^{r*}	20^{j*}	11^{o*}	8^{r*}	5^{l*}	4^{l*}	4^{l*}	1^{*}					
12	57^{x*}	29^{n}	15^{l*}	11^{l*}	6^{m*}	4^{l*}	4^{l*}	4^{l*}	1^{*}				
13	78^{x*}	34^{n}	21^{r}	13^{o*}	10^{l}	6^{e*}	4^{l*}	4^{l*}	4^{l*}	1^{*}			
14	91^{s*}	47^{e}	25^{w}	14^{i*}	11^{h*}	8^{d*}	5^{m*}	4^{l*}	4^{l*}	4^{l*}	1^{*}		
15	124^{e*}	60^{r}	31^{w}	15^{p*}	14^{r}	10^{m}	7^{d*}	5^{l*}	4^{l*}	4^{l*}	4^{l*}	1^{*}	
16	140^{l*}	68^{j}	38^{w}	25^{e}	14^{m*}	13^{r}	8^{m*}	6^{d*}	4^{l*}	4^{l*}	4^{l*}	4^{l*}	1^{*}
17	183^{l*}	68^{s*}	44^{v}	28^{d}	20^{r}	14^{r}	11^{r}	7^{d*}	6^{e*}	4^{l*}	4^{l*}	4^{l*}	4^{l*}
18	207^{x*}	94^{e*}	48^{w*}	34^{d}	24^{d}	16^{r}	12^{m}	10^{r}	6^{m*}	5^{d*}	4^{l*}	4^{l*}	4^{l*}
19	261^{e}	114^{d}	66^{e}	44^{d}	29^{d}	19^{o}	14^{v}	11^{d}	9^{d*}	6^{e*}	5^{l*}	4^{l*}	4^{l*}
20	285^{s*}	145^{e}	75^{d}	52^{d}	30^{m}	25^{r}	15^{l}	14^{l}	10^{m}	8^{d*}	6^{m*}	4^{l*}	4^{l*}
21	352^{e*}	171^{g}	77^{c}	54^{i}	42^{e}	28^{d}	20^{j}	14^{v}	11^{j*}	9^{d*}	7^{d*}	5^{m*}	4^{l*}
22	385^{j*}	200^{c}	77^{l*}	71^{e}	45^{i}	34^{d}	20^{m}	15^{j}	11^{m*}	11^{e}	8^{m*}	6^{d*}	5^{m*}
23	466^{e*}	227^{l}	104^{l*}	75^{d}	51^{d}	38^{d}	24^{j}	15^{j*}	14^{j}	11^{e*}	10^{d}	7^{d*}	6^{e*}
24	510^{x*}	260^{c}	116^{d}	91^{d}	57^{m}	39^{j}	24^{o*}	23^{e}	14^{m*}	14^{e}	11^{m}	8^{m*}	6^{m*}
25	600^{x*}	260^{j}	130^{j}	103^{d}	69^{i}	39^{j}	33^{d}	24^{e}	20^{d}	14^{e}	13^{j}	10^{m}	8^{e*}
26	650^{s*}	260^{j*}	130^{s*}	121^{d}	78^{m}	39^{j}	34^{m}	27^{d}	21^{m}	15^{j}	13^{m}	11^{d}	10^{m}
27	763^{e*}	319^{e*}	167^{e*}	130^{e}	87^{d}	39^{a*}	39^{e}	31^{d}	24^{d}	15^{j*}	14^{j}	12^{m}	11^{e}
28	819^{s*}	372^{u}	189^{d}	153^{d}	91^{m}	56^{e}	39^{e}	36^{d}	25^{m}	22^{e}	14^{m*}	14^{e}	11^{m}
29	950^{e*}	435^{e}	228^{d}	155^{j}	113^{e}	59^{d}	53^{e}	39^{e}	30^{j}	24^{d}	15^{j*}	14^{e*}	13^{d}
30	1020^{x*}	503^{d}	237^{w}	155^{j}	119^{d}	66^{d}	57^{d}	40^{i}	30^{m}	26^{d}	15^{m*}	15^{e}	14^{m}
31	1170^{e}	563^{l}	285^{e}	155^{p}	134^{d}	77^{d}	61^{d}	46^{i}	38^{j}	27^{d}	23^{e}	15^{e*}	14^{j*}
32	1240^{l*}	619^{c}	312^{d}	186^{e}	140^{m}	90^{d}	67^{d}	52^{i}	38^{m}	32^{o}	24^{d}	22^{e}	14^{m*}

Table 3: $t = 3$

$v\backslash k$	5	6	7	8	9	10	11	12	13	14	15	16
5	1^{*}											
6	5^{l*}	1^{*}										
7	9^{l*}	5^{l*}	1^{*}									
8	20^{r*}	7^{j*}	5^{l*}	1^{*}								
9	30^{r*}	12^{l*}	6^{l*}	5^{l*}	1^{*}							
10	51^{r}	20^{r*}	10^{o*}	5^{l*}	5^{l*}	1^{*}						
11	66^{j*}	32^{n*}	17^{j*}	9^{j*}	5^{l*}	5^{l*}	1^{*}					
12	113^{e*}	41^{n}	24^{n}	12^{d*}	8^{j*}	5^{l*}	5^{l*}	1^{*}				
13	157^{n}	66^{n}	30^{n}	19^{r}	10^{j*}	7^{l*}	5^{l*}	5^{l*}	1^{*}			
14	235^{e}	87^{w}	44^{r}	27^{r}	16^{d}	9^{m*}	6^{l*}	5^{l*}	5^{l*}	1^{*}		
15	313^{u}	134^{e}	59^{j}	30^{j}	23^{d}	14^{d}	8^{d*}	5^{l*}	5^{l*}	5^{l*}	1^{*}	
16	437^{e}	178^{d}	90^{e}	30^{a*}	30^{e}	19^{d}	12^{j}	7^{m*}	5^{l*}	5^{l*}	5^{l*}	1^{*}
17	558^{u}	243^{l}	119^{d}	55^{e}	30^{e}	23^{d}	16^{j}	10^{j*}	7^{e*}	5^{l*}	5^{l*}	5^{l*}
18	732^{l}	258^{w}	157^{r}	68^{d}	43^{w}	29^{d}	20^{d}	12^{l}	9^{d*}	6^{m*}	5^{l*}	5^{l*}
19	926^{u}	352^{e}	187^{d}	98^{d}	58^{d}	39^{i}	23^{d}	19^{o}	11^{d}	9^{e*}	6^{l*}	5^{l*}
20	1165^{g}	456^{u}	246^{l}	116^{d}	74^{d}	43^{w}	35^{j}	20^{o}	16^{d}	10^{m*}	8^{d*}	5^{l*}
21	1431^{g}	594^{d}	253^{j}	162^{d}	91^{d}	63^{d}	35^{i}	28^{d}	19^{d}	14^{d}	9^{m*}	7^{d*}
22	1746^{g}	721^{l}	253^{j}	191^{d}	124^{d}	66^{m}	42^{i}	31^{j}	25^{d}	17^{m}	12^{d}	9^{m*}
23	1771^{j*}	871^{l}	253^{l*}	239^{d}	145^{d}	95^{d}	43^{j}	31^{j}	30^{d}	22^{d}	15^{d}	11^{d}
24	2237^{e*}	1035^{l}	357^{l*}	253^{e}	168^{d}	111^{d}	67^{e}	31^{v}	31^{e}	24^{m}	19^{d}	12^{m}
25	2706^{u}	1170^{j}	456^{u}	343^{d}	201^{d}	137^{d}	81^{d}	54^{e}	31^{e}	30^{j}	23^{d}	17^{j}
26	3306^{e}	1170^{j}	585^{u}	369^{d}	249^{d}	143^{d}	94^{d}	55^{d}	46^{j}	30^{m}	27^{d}	19^{m}
27	3906^{u}	1170^{j*}	686^{u}	473^{d}	284^{d}	182^{e}	118^{d}	70^{d}	46^{i}	31^{j}	30^{m}	24^{d}
28	4669^{e}	1489^{e*}	845^{d}	499^{d}	331^{d}	208^{u}	133^{d}	87^{m}	64^{d}	31^{i}	30^{j}	26^{d}
29	5427^{u}	1847^{u}	1005^{d}	620^{j}	379^{d}	264^{e}	157^{d}	94^{d}	70^{d}	53^{e}	30^{j}	30^{e}
30	6239^{l}	2244^{d}	1217^{d}	620^{j}	451^{d}	273^{d}	189^{d}	109^{d}	85^{j}	56^{d}	30^{i*}	30^{m}
31	6852^{j}	2736^{d}	1431^{u}	620^{j}	520^{d}	339^{e}	216^{d}	143^{d}	85^{d}	67^{d}	31^{p*}	30^{e}
32	7843^{l}	3260^{d}	1712^{l}	620^{a}	606^{d}	392^{d}	248^{d}	153^{d}	120^{d}	70^{d}	54^{e}	30^{m*}

Table 4: $t = 4$

$v \backslash k$	6	7	8	9	10	11	12	13	14	15	16
6	1^{*}										
7	6^{l*}	1^{*}									
8	12^{l*}	6^{l*}	1^{*}								
9	30^{r*}	9^{o*}	6^{l*}	1^{*}							
10	50^{r*}	20^{d*}	8^{j*}	6^{l*}	1^{*}						
11	100^{n}	34^{j}	16^{j*}	7^{l*}	6^{l*}	1^{*}					
12	132^{s*}	59^{n}	26^{t*}	12^{o*}	6^{l*}	6^{l*}	1^{*}				
13	245^{e*}	88^{n}	43^{n}	19^{d}	11^{j*}	6^{l*}	6^{l*}	1^{*}			
14	385^{u}	154^{e}	66^{r}	36^{r}	14^{o*}	10^{j*}	6^{l*}	6^{l*}	1^{*}		
15	620^{e}	224^{u}	108^{r}	49^{d}	30^{r}	13^{d*}	9^{j*}	6^{l*}	6^{l*}	1^{*}	
16	840^{l}	358^{e}	118^{l}	79^{e}	41^{d}	22^{d}	12^{m}	8^{l*}	6^{l*}	6^{l*}	1^{*}
17	1277^{e}	506^{r}	208^{e}	94^{u}	58^{d}	36^{j}	17^{d}	11^{j*}	7^{l*}	6^{l*}	6^{l*}
18	1791^{u}	696^{l}	296^{d}	149^{e}	71^{d}	43^{d}	24^{d}	15^{d}	9^{m*}	6^{l*}	6^{l*}
19	2501^{l}	930^{l}	419^{g}	199^{u}	113^{d}	52^{d}	39^{d}	21^{d}	14^{j}	9^{e*}	6^{l*}
20	3297^{g}	1239^{l}	541^{c}	267^{d}	130^{i}	86^{d}	42^{d}	34^{d}	18^{d}	12^{j*}	8^{m*}
21	4322^{g}	1617^{l}	677^{g}	369^{d}	199^{d}	110^{d}	67^{d}	38^{d}	28^{d}	16^{d}	12^{e}
22	5558^{g}	2088^{l}	746^{c}	495^{r}	241^{i}	150^{d}	73^{i}	58^{d}	34^{m}	22^{o}	14^{d}
23	7064^{g}	2647^{l}	759^{c}	622^{d}	357^{c}	194^{d}	86^{j}	69^{i}	52^{d}	31^{d}	19^{d}
24	7084^{s*}	3312^{l}	759^{l*}	748^{d}	408^{i}	266^{d}	86^{w}	79^{i}	59^{m}	44^{d}	24^{o}
25	9321^{e*}	4121^{l}	1116^{l*}	759^{e}	494^{d}	335^{i}	153^{e}	83^{i}	67^{j}	51^{d}	37^{d}
26	11954^{u}	4680^{j}	1543^{u}	1102^{e}	610^{d}	403^{d}	197^{d}	137^{e}	67^{i}	62^{d}	43^{m}
27	15260^{e}	4680^{j}	2090^{d}	1215^{d}	765^{d}	447^{d}	254^{d}	164^{d}	97^{j}	67^{e}	50^{d}
28	19042^{u}	4680^{s*}	2697^{d}	1687^{d}	950^{d}	621^{c}	339^{d}	220^{d}	97^{i}	77^{i}	55^{d}
29	23711^{e}	6169^{e*}	3260^{d}	1901^{d}	1195^{d}	731^{d}	436^{d}	273^{d}	161^{e}	97^{e}	62^{j}
30	28960^{u}	7991^{u}	4186^{d}	2385^{d}	1449^{d}	896^{d}	535^{d}	345^{d}	184^{d}	120^{w}	62^{j}
31	33715^{j}	9966^{d}	5107^{d}	2906^{d}	1761^{l}	1069^{l}	651^{i}	412^{d}	230^{d}	143^{j}	62^{j}
32	36544^{l}	12660^{d}	6430^{d}	3465^{u}	2069^{d}	1263^{l}	744^{i}	496^{i}	293^{d}	191^{d}	62^{a*}

Table 5: $t = 5$

$v \backslash k$	7	8	9	10	11	12	13	14	15	16
7	1^{*}									
8	7^{l*}	1^{*}								
9	16^{l*}	7^{l*}	1^{*}							
10	45^{r*}	12^{j*}	7^{l*}	1^{*}						
11	84^{j}	29^{t*}	10^{j*}	7^{l*}	1^{*}					
12	177^{n}	51^{w}	22^{d*}	9^{j*}	7^{l*}	1^{*}				
13	264^{n}	104^{n}	40^{t}	16^{d*}	8^{l*}	7^{l*}	1^{*}			
14	509^{e}	179^{u}	81^{r}	29^{d}	14^{o*}	7^{l*}	7^{l*}	1^{*}		
15	869^{u}	333^{e}	128^{d}	59^{d}	21^{j}	13^{j*}	7^{l*}	7^{l*}	1^{*}	
16	1489^{e}	522^{u}	219^{r}	95^{d}	46^{d}	19^{j}	12^{j*}	7^{l*}	7^{l*}	1^{*}
17	2234^{u}	829^{r}	305^{u}	156^{r}	70^{d}	36^{j}	17^{d}	11^{j*}	7^{l*}	7^{l*}
18	3511^{e}	1240^{r}	506^{r}	213^{d}	114^{d}	55^{r}	28^{d}	15^{j*}	10^{j*}	7^{l*}
19	5219^{u}	1802^{l}	737^{r}	345^{r}	164^{d}	93^{d}	42^{j}	22^{d}	13^{j*}	9^{l*}
20	7522^{g}	2550^{l}	1049^{r}	492^{r}	254^{r}	126^{d}	71^{d}	32^{d}	19^{d}	12^{m*}
21	10453^{g}	3543^{l}	1466^{c}	691^{g}	358^{g}	196^{c}	94^{d}	58^{d}	27^{d}	17^{j}
22	14290^{g}	4856^{c}	2006^{r}	947^{g}	492^{l}	252^{i}	155^{d}	73^{d}	46^{d}	24^{d}
23	19200^{g}	6533^{l}	2686^{u}	1276^{c}	663^{l}	370^{l}	200^{u}	117^{d}	61^{d}	38^{d}
24	25481^{g}	8630^{l}	3260^{u}	1693^{d}	883^{c}	450^{i}	282^{u}	146^{d}	94^{d}	51^{m}
25	31597^{u}	11317^{c}	3951^{u}	2035^{d}	1160^{l}	647^{g}	329^{u}	203^{d}	119^{d}	82^{d}
26	40918^{e}	14635^{l}	5067^{e}	2452^{d}	1422^{d}	792^{i}	482^{e}	232^{d}	147^{d}	97^{d}
27	52746^{u}	18703^{l}	6562^{u}	3151^{d}	1642^{d}	1078^{g}	614^{i}	356^{d}	180^{d}	124^{d}
28	68006^{e}	22781^{u}	8469^{d}	3995^{d}	2276^{d}	1209^{d}	794^{d}	411^{i}	272^{d}	137^{d}
29	86749^{u}	26893^{u}	10866^{d}	5241^{d}	2857^{d}	1726^{c}	965^{d}	572^{d}	325^{u}	214^{e}
30	109220^{l}	33062^{e}	13149^{d}	6622^{d}	3732^{d}	2159^{c}	1155^{d}	657^{d}	434^{d}	234^{d}
31	133062^{j}	41010^{u}	17035^{d}	8501^{d}	4758^{d}	2670^{c}	1579^{g}	847^{d}	567^{i}	286^{j}
32	154130^{l}	50743^{u}	21140^{d}	10556^{d}	5862^{c}	3285^{c}	1944^{c}	1087^{i}	709^{d}	286^{w}

Table 6: $t = 6$

$v\backslash k$	8	9	10	11	12	13	14	15	16
8	1^{*}								
9	8^{l*}	1^{*}							
10	20^{j*}	8^{l*}	1^{*}						
11	63^{j*}	15^{l*}	8^{l*}	1^{*}					
12	126^{t}	40^{r*}	12^{o*}	8^{l*}	1^{*}				
13	297^{n}	79^{n}	30^{d}	11^{j*}	8^{l*}	1^{*}			
14	474^{j}	183^{e}	58^{t}	22^{d*}	10^{j*}	8^{l*}	1^{*}		
15	983^{e}	325^{d}	132^{r}	45^{d}	18^{d*}	9^{l*}	8^{l*}	1^{*}	
16	1806^{u}	636^{d}	232^{d}	99^{r}	28^{d}	16^{o*}	8^{l*}	8^{l*}	1^{*}
17	3295^{e}	1093^{r}	407^{r}	163^{d}	72^{d}	26^{j}	15^{j*}	8^{l*}	8^{l*}
18	5354^{u}	1775^{c}	659^{c}	283^{r}	122^{d}	50^{d}	24^{j}	14^{j*}	8^{l*}
19	8865^{e}	2800^{l}	1048^{r}	448^{r}	210^{d}	90^{d}	42^{j}	19^{d*}	13^{j*}
20	13838^{l}	4277^{c}	1607^{r}	693^{r}	327^{r}	164^{r}	60^{d}	34^{d}	17^{j*}
21	20664^{g}	6388^{l}	2407^{c}	1042^{g}	496^{c}	229^{d}	131^{e}	50^{d}	28^{d}
22	30045^{g}	9292^{c}	3509^{c}	1526^{c}	726^{g}	372^{c}	183^{d}	94^{d}	40^{d}
23	42944^{g}	13300^{l}	5039^{l}	2186^{c}	1047^{l}	539^{l}	291^{l}	144^{d}	76^{d}
24	60164^{g}	18662^{l}	7073^{c}	3086^{l}	1476^{l}	760^{g}	414^{g}	235^{l}	113^{d}
25	83017^{l}	25770^{c}	9783^{c}	4275^{l}	2051^{l}	1059^{l}	579^{g}	324^{d}	192^{d}
26	112252^{l}	35103^{l}	12896^{l}	5834^{l}	2803^{l}	1449^{c}	743^{i}	454^{r}	243^{d}
27	150647^{l}	47150^{c}	17597^{l}	7856^{l}	3784^{c}	1955^{c}	1073^{l}	618^{c}	367^{g}
28	197976^{l}	62562^{l}	23571^{l}	10453^{c}	5039^{c}	2613^{c}	1379^{i}	827^{l}	446^{i}
29	259931^{l}	82094^{l}	31097^{l}	13737^{l}	6628^{c}	3441^{l}	1890^{l}	1090^{i}	656^{l}
30	337223^{l}	106616^{l}	40540^{l}	17879^{l}	8641^{l}	4495^{l}	2473^{c}	1427^{l}	741^{i}
31	430492^{j}	137079^{l}	52297^{l}	23042^{c}	11144^{c}	5799^{c}	3197^{c}	1842^{i}	1078^{i}
32	532248^{l}	174784^{l}	66824^{l}	29423^{c}	14252^{c}	7418^{g}	4097^{c}	2342^{i}	1190^{i}

Table 7: $t = 7$

$v\backslash k$	9	10	11	12	13	14	15	16
9	1^{*}							
10	9^{l*}	1^{*}						
11	25^{r*}	9^{l*}	1^{*}					
12	84^{t*}	18^{l*}	9^{l*}	1^{*}				
13	185^{t}	52^{t*}	15^{d*}	9^{l*}	1^{*}			
14	482^{e}	121^{u}	40^{d}	13^{d*}	9^{l*}	1^{*}		
15	790^{j}	300^{u}	81^{t}	30^{d*}	12^{d*}	9^{l*}	1^{*}	
16	1773^{e}	553^{d}	209^{r}	65^{d}	24^{d*}	11^{d*}	9^{l*}	1^{*}
17	3499^{u}	1160^{r}	393^{d}	153^{r}	44^{d}	20^{d*}	10^{l*}	9^{l*}
18	6794^{e}	2083^{c}	717^{r}	280^{r}	107^{d}	34^{d}	18^{o*}	9^{l*}
19	11827^{u}	3579^{r}	1227^{l}	487^{r}	192^{d}	76^{d}	31^{d}	17^{d*}
20	20692^{e}	5934^{c}	2055^{l}	814^{r}	355^{r}	150^{d}	57^{d}	26^{d}
21	33718^{g}	9499^{l}	3313^{g}	1321^{c}	582^{g}	274^{c}	96^{d}	49^{d}
22	52674^{g}	14900^{l}	5186^{g}	2072^{c}	915^{g}	437^{l}	219^{l}	71^{d}
23	80027^{g}	22699^{g}	7917^{l}	3182^{l}	1410^{g}	674^{l}	316^{d}	160^{d}
24	119064^{l}	33830^{c}	11828^{c}	4765^{l}	2118^{l}	1013^{l}	517^{l}	254^{d}
25	172071^{l}	49556^{l}	17331^{c}	7000^{g}	3118^{l}	1498^{l}	765^{l}	409^{l}
26	246965^{l}	71206^{c}	24924^{c}	10079^{c}	4504^{c}	2166^{c}	1110^{l}	597^{l}
27	347268^{l}	100709^{l}	34976^{l}	14320^{l}	6400^{c}	3086^{l}	1583^{c}	853^{l}
28	480708^{l}	140394^{c}	49017^{l}	19988^{g}	8960^{c}	4329^{c}	2221^{c}	1202^{g}
29	650404^{l}	193066^{l}	67625^{l}	27561^{c}	12364^{l}	5992^{c}	3080^{c}	1669^{l}
30	879517^{l}	262146^{l}	92034^{l}	37494^{l}	16849^{c}	8176^{l}	4213^{l}	2252^{i}
31	1174351^{l}	351807^{l}	123856^{l}	50435^{c}	22687^{l}	11018^{l}	5685^{c}	3085^{c}
32	1530641^{l}	467414^{l}	164722^{l}	67117^{c}	30228^{c}	14697^{l}	7601^{l}	4130^{i}

Table 8: $t = 8$

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